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ON THE KINETIC THEORY OF WAVE PROPAGATION IN RANDOM MEDIA

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CONTENTS

This paper considers the theory of the multiple scattering of waves in extensive random media. The classical theory of wave propagation in random media is discussed with reference to its practical limitations, and in particular to the inability of the lowest order approximation to the Bethe–Salpeter equation, which describes the propagation of correlations, to account for conservation of energy. An alternative *kinetic* theory is formulated, based on the theory of energy transfer processes in random media. The proposed theory satisfies conservation of energy and the Second Law of Thermodynamics. It is illustrated by a consideration of three problems each of which is difficult or impossible to treat by classical scattering theory. These involve the transmission of energy through a slab of random medium; the scattering theory of geometrical optics; and scattering by a randomly inhomogeneous half-space.

1. Introduction

The basis of the modern kinetic theory of gases originated with the work of Maxwell and Boltzmann in the nineteenth century. The fundamental equations of the subject were derived by means of apparently crude phenomenological considerations of the energy and momentum exchanges between the individual molecules of a gas during a classical binary collision. Yet the theory was remarkably successful in predicting the properties of many of the common gases.

A large body of problems of a similar nature occurs in the general area of wave propagation phenomena. For example the presence of nonlinear terms in a wave equation can produce *resonant interactions* between 'wave packets' (Phillips 1960). The net result is generally one in which there is a transfer of energy between the interacting modes together, possibly, with the

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creation of new wave packets. Success in describing such processes by means of equations of the Boltzmann type, however, has been limited to cases where the hypothesis of 'molecular chaos' is a reasonable one. This hypothesis asserts that interacting wave packets are statistically independent, a condition which imposes severe limitations on the nonlinear wave system. In particular it requires that the wavenumber-frequency relation for the propagating wave modes be strongly nonlinear so that, because of the resulting differences in *group velocities*, interacting waves spend only a relatively short portion of their respective lifetimes actually interacting with one another (Benney & Saffman 1966; Bretherton 1969). Clearly this rules out of consideration a great many problems of practical importance such as especially those in acoustics.

There are many systems, however, in which the interaction between the admissible wave modes is not described by a nonlinear term in the wave equation. We have in mind a large class of phenomena normally termed *scattering*. Deviations in the properties of a wave-bearing medium from their otherwise constant values transfer a portion of the energy of an incident wave into a whole spectrum of scattered waves. When the dimensions of the region containing these deviations are large, so that cumulative multiple scattering effects are likely to be important, it no longer makes sense to attempt to describe the propagation processes by means of a naive scattering theory such as the *Born approximation*.

When, further, the deviations in the properties of the medium may be regarded as *random*, for example in situations where knowledge of the inhomogeneities is restricted to a statistical description alone, then the scattered waves are also random and multiple scattering of one of these waves results in a transfer of energy to yet other random modes. If we can ensure that the distance travelled by a scattered wave packet during its lifetime is large compared with the *correlation scale* of the inhomogeneities of the medium, then for most of its lifetime a wave packet is propagating through parts of the medium with which it is uncorrelated. In other words, we have essentially a state of affairs in which a random wave field is interacting with a random medium, and the basic interacting elements are *statistically independent*. This constitutes an analogue of the classical notion of molecular chaos. It would seem to be desirable, therefore, to inquire into the possibility of developing a kinetic theory to describe the scattering processes in such an extensive random medium.

The object of the present paper is to propose such a theory.

Section 2 consists of a critique of the classical theory of wave propagation in random media, together with an outline of the theory of energy transfer processes which has been considered in detail in an earlier report (Howe 1973). This preliminary material in conjunction with a simple qualitative argument enables us to write down the desired kinetic equation (§3). The same equation can also be derived by means of a more elaborate multiple scale procedure. That analysis is discussed separately in a companion paper (Howe 1972*a*) since its inclusion here would serve only to obscure the relatively simple physical arguments involved. Then follows a general discussion of the principal properties of the kinetic equation (§4). In particular the principle of conservation of total wave energy is established and an appropriate form of Boltzmann's H-theorem deduced. Finally, the theory is illustrated in §5 by a consideration of three simple problems each of which is difficult or impossible to treat by classical methods. These problems involve the transmission of energy through a slab of random medium; the scattering theory of geometrical optics; and reflexion of an incident wave from an inhomogeneous half-space. It should be remarked that Budden & Uscinski have recently discussed a series of closely related problems with regard to the propagation of electromagnetic radiation through a medium con-

taining weak random irregularities in refractive index (Budden & Uscinski 1970, 1971, 1972). Their work includes a detailed analysis of multiple scattering and is essentially complementary to the material presented here.

The present analysis is restricted to problems in which the inhomogeneities of the medium are *random functions of position alone*. This simplifies the arguments and also eliminates complications arising from spectral line broadening effects. Actually it is a relatively straightforward matter to include randomness in time in certain circumstances, and this indeed has important applications to such problems as spectral broadening of sound propagating through a turbulent jet (Howe 1972*b*).

2. Wave propagation in random media; energy exchange processes

2.1.

In this section basic results regarding the propagation of waves in a conservative random medium are reviewed. The analysis is restricted to a consideration of a rather wide class of linear *scalar* wave equations. However, it should be borne in mind that the final results can readily be extended to cover more complicated systems specified by a vector field variable.

In an *extensive* random medium it is essential to recognize that the cumulative effects of even quite small random inhomogeneities in the properties of the medium can rapidly destroy any *coherence* initially associated with an incident wave. Therefore an attempt to specify the effects of the inhomogeneities by means of a perturbation expansion will require a large number of terms in order to ensure the elimination of an initially coherent wave. The problem is actually so complicated that the calculation of the evolution of the complete wave field by such a procedure is impracticable. It is feasible only if it is possible to sum all the terms in the perturbation expansion. Now the expansion describes multiple scattering processes and must contain *secular* terms (in fact an infinite number); in general there is no guarantee that it can be summed in closed form. In order to effect a summation it is generally necessary to confine attention to a limited description of the wave field.

This is in terms of the *ensemble average field*, obtained by averaging over an assembly of statistically equivalent random media. When the formal perturbation expansion is averaged it becomes clear that an infinite subseries of non-secular terms M, say, can be extracted, and the expansion expressed as an infinite geometrical progression in powers of M and the free space Green function, successive terms of which are of increasing secularity. This new expansion, however, can be recognized as a formal perturbation solution of a *renormalized*, *non-random* wave equation for the ensemble average field (Frisch 1968).

It is natural therefore to partition the wave field, ϕ, say, into an ensemble average, or *coherent*, component $\overline{\phi}$, and a fluctuating part ϕ' which describes the deviation of ϕ from the mean $\overline{\phi}$ in a given realization of the medium, i.e.

$$\phi = \overline{\phi} + \phi'. \tag{2.1}$$

The renormalized equation describes the propagation of the mean wave field $\overline{\phi}$. It possesses physical characteristics which are not present in the original wave equation. The most important is the presence of an energy dissipation term which accounts for the *irreversible* transfer of energy from the mean field to the random waves ϕ'. Secondly, the *group velicity* of a mean field wave packet is less than that for propagation in free space. This is because in a particular realization

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the wave will be disturbed from its free space geodesic path of propagation by the inhomogeneities, and consequently on average takes longer to cover a specified distance.

The equation for the mean field is familiar in Quantum field theory where it is known as the *Dyson equation* (Dyson 1949). Various approximations to the Dyson equation have been obtained independently by several authors (e.g. Meecham 1961; Bourret 1962; Keller 1964; Frisch 1968; Howe 1971*a*). Actually if a small dimensionless parameter ϵ is introduced as a measure of the magnitude of the random inhomogeneities, then the leading term in the non-secular subseries M is of order ϵ^2, and the approximation to the Dyson equation obtained by neglecting the remaining terms in M is called the *binary interaction* equation. This is normally the only version of the Dyson equation amenable to analytic investigation.

To be more specific consider a scalar wave equation represented symbolically by

$$L\phi = G\phi. \tag{2.2}$$

We adopt the convention that in this equation terms involving the random inhomogeneities of the medium are all contained in the linear operator G, and therefore that the linear operator L describes propagation through a uniform medium ('free space'). Thus G represents an $O(\epsilon)$ modification of the free space wave equation $L\phi = 0$. In particular, since the random terms describe parameter fluctuations about their constant mean values, this implies that the ensemble average of G, namely $\overline{G}$, vanishes identically.

The *binary interaction approximation* to the Dyson equation is then

$$L\overline{\phi} = \overline{GL^{-1}G\overline{\phi}}, \tag{2.3}$$

where L^{-1} is the retarded time Green function operator inverse to L (see, for example, Howe 1971*a*). This equation may be expressed as the pair

$$L\overline{\phi} = \overline{G\phi'}; \quad L\phi' = G\overline{\phi}. \tag{2.4a, b}$$

When the formal solution of (2.4*b*) is inserted into the first of these equations we obtain precisely the binary interaction equation (2.3) Now (2.4*b*) is the *local Born approximation* describing the *generation* of the random waves ϕ' by the interaction between the inhomogeneities and the mean field $\overline{\phi}$. Of course that equation neglects multiple scattering of the random waves so generated, and is therefore unsuitable for determining the random field *per se*. The formal use of the solution ϕ' in equation (2.4*a*) therefore throws light on the physical significance of the binary interaction approximation. The mean value on the right of (2.4*a*) is in fact a *cross-correlation product* of the random operator $G(\boldsymbol{x})$ and the random field ϕ' at the point $\boldsymbol{x}$, say. Thus only those constituent waves of ϕ' which were scattered within a correlation scale l of the point $\boldsymbol{x}$ can contribute to the mean value. Here l refers to the correlation scale of the random inhomogeneities. Hence in using (2.4*b*) to determine this correlation product it is implicitly assumed that multiple scattering is not important *only* over distances of order l, and that the *direct* random field given by (2.4*b*) is large compared with the contribution from the random waves which are re-scattered within this distance l from $\boldsymbol{x}$. The direct field gives an $O(\epsilon^2)$ contribution, whereas these remaining terms are at least $O(\epsilon^3)$.

2.2.

The continuous generation of the random field ϕ' implies that in many situations involving extensive inhomogeneous regions a substantial portion of the total energy is actually contained in

the incoherent waves. The lack of coherence suggests that it is now appropriate to try to develop an equation for $\overline{\phi^2}$ – this being a direct measure of the energy density of the *whole* field. A formal analysis analogous to that outlined above (see Frisch 1968) for the derivation of the Dyson equation gives a renormalized equation *not* for $\overline{\phi^2}$, however, but for the two-point correlation product $\overline{\phi(\boldsymbol{x}, t)\,\phi(\boldsymbol{X}, T)}$ (t, T denoting the respective times), and is referred to as the *Bethe–Salpeter equation* (Salpeter & Bethe 1951). Under certain simplifying circumstances that equation can be reduced to an equation for $\overline{\phi^2}$, and thence to an *energy transport equation*. In particular it is necessary to assume that the correlation scale l is small compared to the wavelength. Naturally this restriction quite often precludes a consideration of problems of practical importance: significant effects of scattering are normally expected in cases where these lengths are of the same order of magnitude.

As in the case of the Dyson equation it is generally only the lowest order approximation to the Bethe–Salpeter equation (in which only the first non-trivial term of the non-secular subseries is retained) which is suitable for analytic consideration. But even this reduced form is often difficult to treat in any but the simplest type of problem, and then usually requires further simplifying assumptions, such as that mentioned above, to be imposed. A rather more serious drawback has been pointed out by Frisch (1968). In considering the energy radiated from an acoustic point source in an infinite random medium, Frisch shows that in the lowest order non-trivial approximation *the Bethe–Salpeter equation violates the principle of conservation of energy*. Apparently energy conservation can be assured only if *all* of the non-secular terms are retained. It seems, therefore, that the situation can only be rectified by means of an approximation to the complete Bethe–Salpeter equation which both conserves energy *and* is mathematically tractable.

It is precisely because of these difficulties that the author was led to an alternative consideration of the problem of determining $\overline{\phi^2}$. This alternative approach constitutes the core of the present paper. Admittedly the proposed theory is itself only expected to be valid for a certain combination of the ranges of values of the parameters describing the system, but we shall see that this includes many cases of practical importance.

2.3.

First we develop a notation which will enable us to cover a wide range of scalar wave propagation problems.

Define differential operators Q_j, H_j, K_j, typically by

$$Q_j = l_m^a\, l_n^b\, l_p^c \ldots \frac{\partial}{\partial x_q}\frac{\partial}{\partial x_r}\frac{\partial}{\partial x_s}\ldots, \tag{2.5}$$

where the l_m^a are unit vectors (possibly absent) which would be required in a description of a system with anisotropic mean properties. The suffix j is to be regarded as a *label* specifying a particular operator and *not* as the number of differential coefficients, etc., involved in that operator. Two operators such as H_j, K_j with the same suffix are distinct, but a product of the form $H_j \phi K_j$, involving two such operators with the same suffix, is assumed to be a real, scalar operator.

The *adjoint* operator is defined typically by

$$\tilde{Q}_j = l_m^a\, l_n^b\, l_p^c \ldots \left(-\frac{\partial}{\partial x_q}\right)\left(-\frac{\partial}{\partial x_r}\right)\left(-\frac{\partial}{\partial x_s}\right)\ldots. \tag{2.6}$$

A large class of scalar propagation problems in conservative random media can now be defined by means of the *Lagrangian density*

$$\mathscr{L} = \sum_j \tfrac{1}{2}A_j(1+\xi_j)\left(Q_j\frac{\partial\phi}{\partial t}\right)^2 - \sum_j B_j\,(1+\zeta_j)\,(H_j\phi)\,(K_j\phi)$$
$$= \text{kinetic energy per unit volume} - \text{potential energy per unit volume}. \tag{2.7}$$

The reason for following an approach based on a Lagrangian density is that it enables one to make rather precise statements regarding energy conservation, etc.

In this definition $\xi_j(\boldsymbol{x})$, $\zeta_j(\boldsymbol{x})$ are assumed to be *stationary random functions of position* $\boldsymbol{x}$ in the medium, with $|\xi_j|$, $|\zeta_j| \simeq O(\epsilon)$ ($\epsilon \ll 1$). They represent the deviations of the physical parameters of the medium from their otherwise *constant* mean values A_j, B_j respectively, so that $\bar{\xi}_j \equiv 0$, $\bar{\zeta}_j \equiv 0$, where an overbar denotes an ensemble average.

The wave equation is obtained by requiring

$$\delta\iint_{\mathrm{R}} \mathscr{L}\left(Q_j\frac{\partial\phi}{\partial t}, H_j\phi, K_j\phi\right)\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$

to vanish for variations which are zero on the boundaries of the region R of space-time. This leads to a wave equation of the form

$$L\phi = G\phi \tag{2.8}$$

considered earlier (see Gelfand & Fomin 1963), where the operators L, G are given by

$$\left.\begin{aligned} L &= \sum_j\{-A_j\tilde{Q}_jQ_j\,\partial^2/\partial t^2 - B_j(\tilde{K}_jH_j + K_j\tilde{H}_j)\},\\ G &= \sum_j\{A_j\tilde{Q}_j\xi_jQ_j\,\partial^2/\partial t^2 + B_j(\tilde{K}_j\zeta_jH_j + \tilde{H}_j\zeta_jK_j)\}.\end{aligned}\right\} \tag{2.9}$$

Now for a system specified by a Lagrangian density such as (2.7) the energy equation may be derived by multiplying the wave equation (2.8) by $\partial\phi/\partial t$ and rearranging the terms as a time derivative plus a divergence. This is possible because the system is non-dissipative. The method may be extended to obtain equations describing the transfer of energy from the mean field to the random field.

To do this first take the ensemble average of (2.8), obtaining

$$L\bar{\phi} = \overline{G\phi'}; \tag{2.10a}$$

subtract this from (2.8):

$$L\phi' - [G\phi' - \overline{G\phi'}] = G\bar{\phi}. \tag{2.10b}$$

If the first of these equations is multiplied by $\partial\bar{\phi}/\partial t$ and rearranged, we obtain the following equation for the *coherent* field energy:

$$\frac{\partial}{\partial t}\left\{\sum_j \tfrac{1}{2}A_j\left(Q_j\frac{\partial\bar{\phi}}{\partial t}\right)^2 + \sum_j B_j(H_j\bar{\phi})\,(K_j\bar{\phi})\right\} + \mathrm{div}\{\} = -\frac{\partial\bar{\phi}}{\partial t}\overline{G\phi'}, \tag{2.11}$$

where the second term on the left is a divergence whose form is known in principle, but is not important in the present discussion. Similarly, if (2.10b) is multiplied by $\partial\phi'/\partial t$ and averaged we obtain the energy equation for the random field:

$$\frac{\partial}{\partial t}\left\{\sum_j \tfrac{1}{2}A_j\overline{(1+\xi_j)\left(Q_j\frac{\partial\phi'}{\partial t}\right)^2} + \sum_j B_j\overline{(1+\zeta_j)\,(H_j\phi')\,(K_j\phi')}\right\} + \mathrm{div}\{\} = -\overline{\frac{\partial\phi'}{\partial t}G\bar{\phi}}. \tag{2.12}$$

When these equations are added together we recover the ensemble average of the energy equation for the whole system. Note that in this case the terms on the right of (2.11), (2.12) combine to give the time derivative of the mean interaction energy density, viz.

$$\sum_j \frac{\partial}{\partial t}\left\{A_j \xi_j \overline{\left(Q_j \frac{\partial \phi'}{\partial t}\right)}\left(Q_j \frac{\partial \overline{\phi}}{\partial t}\right) + B_j \zeta_j \overline{[(H_j \phi')\,(K_j \overline{\phi}) + (H_j \overline{\phi})\,(K_j \phi')]}\right\}$$

together with a divergence term.

The occurrence of the cross-product terms on the right-hand sides of the energy equations is responsible for the irreversible transfer of energy from the mean field $\overline{\phi}$ to the random field ϕ' (Howe 1973). Observe that terms involving the random inhomogeneities ξ_j, ζ_j occur also on the left-hand side of (2.12); they account for *multiple scattering of the random waves*. The transfer of energy to the random field involves only first scattering of the mean field, and the results of a detailed analysis of that transfer process will now be recalled (for further details see Howe 1973).

Actually it is convenient to express the random operator G formally as a sum of terms of the form

$$G = \sum_j \alpha_j \{\tilde{S}_j \eta_j T_j + \tilde{T}_j \eta_j S_j\}, \tag{2.13}$$

where

$$S_j \equiv S_j\left(\frac{\partial}{\partial \boldsymbol{x}}, \frac{\partial}{\partial t}\right), \quad T_j \equiv T_j\left(\frac{\partial}{\partial \boldsymbol{x}}, \frac{\partial}{\partial t}\right), \tag{2.14}$$

are differential operators analogous to (2.5), η_j is a random inhomogeneity such as ξ_j or ζ_j, and α_j is a constant.

Cross-correlation products are then defined by

$$R_{ij}(\boldsymbol{x} - \boldsymbol{X}) = \overline{\eta_i(\boldsymbol{x})\,\eta_j(\boldsymbol{X})}, \tag{2.15}$$

and exist in this form since the η_j are stationary random processes. They are $O(\epsilon^2)$ quantities, but not necessarily even functions of $\boldsymbol{x} - \boldsymbol{X}$. The corresponding *cross-spectra* are given by means of the Fourier space transform:

$$\Phi_{ij}(\boldsymbol{\kappa}) = \frac{1}{(2\pi)^\nu} \int_{-\infty}^{\infty} R_{ij}(\boldsymbol{x})\, \mathrm{e}^{-\mathrm{i}\boldsymbol{\kappa}\cdot\boldsymbol{x}} \mathrm{d}\boldsymbol{x}, \tag{2.16}$$

where ν is the number of space coordinates involved in the problem.

Now set

$$P_{\mathrm{M}} = -\frac{\partial \overline{\phi}}{\partial t} \overline{G\phi'}, \tag{2.17}$$

the rate of change of mean field energy per unit volume resulting from the interaction of the field with the random inhomogeneities, and

$$P_{\mathrm{R}} = -\overline{\frac{\partial \phi'}{\partial t} G\overline{\phi}}, \tag{2.18}$$

the ensemble average of the corresponding rate for the random field. These expressions are characterized by two sets of space and time scales. The local variations in P_{M}, P_{R} are caused by the local variations of the phases of the various components of the *mean* wave field, and are unimportant as far as the question of energy transfer is concerned – they are associated with the oscillation of the field energy between the kinetic and potential forms. Much larger scales of variation, however, are associated with the distances and times over which significant changes in

the *amplitude* of the mean field occur. Examination of the binary interaction equation for the mean field reveals that these scales are $O(1/\epsilon^2)$. It is clear, therefore, that changes in the net rate of energy transfer are not related to the local phase flucutations of $P_{\rm M}$, $P_{\rm R}$, but to the long term $O(1/\epsilon^2)$ scales. This means that in working out the net rates, $P_{\rm M}$, $P_{\rm R}$ should be averaged over distances and times large compared with the scales corresponding to the local fluctuations of the mean field (i.e. period and wavelength), but small in comparison with the scales over which significant changes occur in the parameters of the mean field.

We shall use angle brackets, $\langle\,\rangle$, to denote such an average. Then if the *binary interaction approximation* to the mean field, and the *local Born approximation* to the random field are used, it is readily deduced that

$$\langle P_{\rm M}\rangle = -\langle P_{\rm R}\rangle \leqslant 0, \tag{2.19}$$

at least correct to $O(\epsilon^2)$, showing that the random field grows irreversibly at the expense of the coherent field, as expected.

More explicitly, consider a mean field $\overline{\phi}$ defined by a sum of *propagating modes*

$$\overline{\phi} = \sum_p a_p(\boldsymbol{x}, t)\exp\{\mathrm{i}(\boldsymbol{k}_p.\,\boldsymbol{x} - \omega_p t)\}, \tag{2.20}$$

with $a_{-p} = a^*{}_p$, $\boldsymbol{k}_{-p} = -\boldsymbol{k}_p$, $\omega_{-p} = -\omega_p$. In this expression the constituent waves are propagating in the sense that $\boldsymbol{k}_p$, ω_p are assumed to be *real* and to satisfy the free space dispersion relation obtained from equation (2.8) by setting $G = 0$ and putting $\phi = \exp\{\mathrm{i}(\boldsymbol{k}_p.\,\boldsymbol{x} - \omega_p t)\}$, namely $L(\boldsymbol{k}_p, \omega_p) = 0$. Actually examination of the non-random form of the wave equation reveals that

$$L(\boldsymbol{k}, \omega) \equiv P(\boldsymbol{k})\,\omega^2 - Q(\boldsymbol{k}), \tag{2.21}$$

where P, Q are *even*, *non-negative*, *polynomial* functions of $\boldsymbol{k}$. Further, the amplitude factors $a_p(\boldsymbol{x}, t)$ change significantly only over distances and times of order $1/\epsilon^2$.

Let us now introduce the following shorthand notation. For real wavenumber vectors $\boldsymbol{K}$, $\boldsymbol{k}_p$ and frequency ω_q define the quantity $\{\boldsymbol{k}_p, \boldsymbol{K}\}^q_j$ by

$$\{\boldsymbol{k}_p, \boldsymbol{K}\}^q_j \equiv S_j(-\mathrm{i}\boldsymbol{K}, \mathrm{i}\omega_q)\,T_j(\mathrm{i}\boldsymbol{k}_p, -\mathrm{i}\omega_q) + S_j(\mathrm{i}\boldsymbol{k}_p, -\mathrm{i}\omega_q)\,T_j(-\mathrm{i}\boldsymbol{K}, \mathrm{i}\omega_q). \tag{2.22}$$

It is easy to show that $\{\boldsymbol{k}_p, \boldsymbol{K}\}^q_j$ is a real function of its arguments and that

$$\{\boldsymbol{k}_p, \boldsymbol{K}\}^q_j \equiv \{-\boldsymbol{k}_p, -\boldsymbol{K}\}^q_j \equiv \{\boldsymbol{K}, \boldsymbol{k}_p\}^q_j. \tag{2.23}$$

With these preliminaries we can now state the result derived by Howe (1973) for the net rate of transfer of energy per unit volume of the medium. Howe shows that uniformly to $O(\epsilon^2)$

$$\langle P_{\rm M}\rangle = -\langle P_{\rm R}\rangle = -2\pi\sum_{p>0}\int |\omega_p|\,|a_p(\boldsymbol{x}, t)|^2\,\Phi(\boldsymbol{k}_p, \boldsymbol{K})\,\delta\{L(\boldsymbol{K}, \omega_p)\}\,\mathrm{d}\boldsymbol{K}, \tag{2.24}$$

where the integration is over all wavenumbers $\boldsymbol{K}$ of the first scattered random modes, which are responsible for the transfer of energy from the mean field to the random field. In this expression

$$\Phi(\boldsymbol{k}_p, \boldsymbol{K}) \equiv \Phi(\boldsymbol{K}, \boldsymbol{k}_p) = \sum_{i,j}\alpha_i\alpha_j\{\boldsymbol{k}_p, \boldsymbol{K}\}^p_i\,\{\boldsymbol{k}_p, \boldsymbol{K}\}^p_j\,\Phi_{ij}(\boldsymbol{K} - \boldsymbol{k}_p), \tag{2.25}$$

which can be shown to be *non-negative*, and $\delta(x)$ is the one dimensional Dirac δ-function.

The result (2.24) illustrates in a precise manner the mechanism responsible for the mean energy transfer rate. In the first place the presence of the δ-function implies that only the *propagating* random wave modes play a significant rôle in the transfer process. Secondly, the integrand

in (2.24) is non-negative so that the presence of the minus sign demonstrates unambiguously that $\langle P_{\mathrm{M}} \rangle$ is negative, i.e. that the net flow of energy is indeed from the mean field to the random field. Of course this was to be expected; the state of disorder of the system can only be increased by the presence of the random inhomogeneities and the corresponding increase in entropy is guaranteed by the Second Law of Thermodynamics.

Equation (2.24) is the main result of this section. It will be used in the following section in conjunction with a simple qualitative argument to derive a kinetic equation for an ensemble of random wave packets.

3. The kinetic equation for an assembly of random wave packets

We now present an essentially heuristic analysis of multiple scattering in a random medium. An alternative treatment based on a formal multiple scale argument and leading to the same conclusions is described in the companion paper (Howe 1972*a*).

The angle bracket averaging procedure leading to equation (2.24) actually allows us to consider the decay of an individual mode of the mean field (Howe 1973). In particular for a plane wave

$$\overline{\phi}_k = a(\boldsymbol{x}, t) \exp \mathrm{i}\{(\boldsymbol{k}.\boldsymbol{x} - \omega t)\} + \text{c.c.}, \tag{3.1}$$

(where c.c. denotes the complex conjugate of the preceding expression), we have from (2.24), in an obvious notation,

$$\langle P^k_{\mathrm{M}} \rangle = -\langle P^k_{\mathrm{R}} \rangle = -2\pi |\omega| |a(\boldsymbol{x}, t)|^2 \int \Phi(\boldsymbol{k}, \boldsymbol{K}) \delta\{L(\boldsymbol{K}, \omega)\} \, \mathrm{d}\boldsymbol{K}. \tag{3.2}$$

As a first step in deriving the kinetic equation we shall now obtain the energy equation for $\overline{\phi}_k$. To do this it is convenient to write the appropriate mean field energy equation (2.11) in the primitive form

$$\left\langle \frac{\partial \overline{\phi}_k}{\partial t} L \overline{\phi}_k \right\rangle = -\langle P^k_{\mathrm{M}} \rangle. \tag{3.3}$$

obtained by multiplying equation (2.10*a*) for the complete mean field by $\partial \overline{\phi}_k / \partial t$ and isolating the mode $\overline{\phi}_k$ by means of the angle bracket averaging procedure. Substituting (3.1) into the left-hand side of this equation gives

$$\left(\frac{\partial a^*}{\partial t} + \mathrm{i}\omega a^* \right) L\left(\boldsymbol{k} - \mathrm{i}\frac{\partial}{\partial \boldsymbol{x}}, \omega + \mathrm{i}\frac{\partial}{\partial t} \right) a + \text{c.c.} = -\langle P^k_{\mathrm{M}} \rangle. \tag{3.4}$$

Next expand the operator L in this expression in powers of the derivatives and retain only the first non-trivial terms, since $a(\boldsymbol{x}, t)$ varies slowly on the scales $1/k$, $1/\omega$. Then correct to lowest order in these derivatives

$$\omega \frac{\partial L}{\partial \omega} \frac{\partial |a|^2}{\partial t} - \omega \frac{\partial L}{\partial k_j} \frac{\partial |a|^2}{\partial x_j} = \langle P^k_{\mathrm{M}} \rangle. \tag{3.5}$$

To interpret this result note that from the left-hand side of equation (2.11), the energy density of the mean wave averaged over a wavelength is just

$$\begin{aligned} E(\boldsymbol{x}, t) &= [P(\boldsymbol{k}) \, \omega^2 + Q(\boldsymbol{k})] \, |a|^2 + O(\epsilon^2) \\ &= \omega \frac{\partial L}{\partial \omega} |a|^2 + O(\epsilon^2), \end{aligned} \tag{3.6}$$

by (2.21). Using this in (3.5) and also substituting for $\langle P^k_{\mathrm{M}}\rangle$ from (3.2), we finally deduce that E satisfies

$$\frac{\partial E}{\partial t}+\frac{\partial \omega}{\partial k_j}\frac{\partial E}{\partial x_j}=\frac{-2\pi E}{|\partial L/\partial \omega|}\int \Phi(\boldsymbol{k},\boldsymbol{K})\,\delta\{L(\boldsymbol{K},\omega)\}\,\mathrm{d}\boldsymbol{K}. \tag{3.7}$$

Here

$$\frac{\partial \omega}{\partial k_j}=-\frac{\partial L}{\partial k_j}\Big/\frac{\partial L}{\partial \omega}$$

is the *group velocity* of the mean field wave packet $\overline{\phi}_k$.

In the absence of scattering (3.7) reduces to the well-known result that wave energy propagates at the group velocity. Since the right-hand side of (3.7) is negative the description is now one in which the mean wave is gradually attenuated by scattering off the random inhomogeneities.

It has already been remarked that the contributions to the integral in (3.7) come solely from the propagating wavenumbers $\boldsymbol{K}$, and in this case from those $\boldsymbol{K}$ which can radiate at the frequency ω of the coherent wave $\overline{\phi}_k$. This is just the scattering process responsible for the generation of the random field of wave packets.

Consider the evolution of one of these packets. Suppose that the damping scale associated with the decay of coherence ($\sim O(1/\epsilon^2)$) is *large* compared with the *correlation* scale l of the inhomogeneities. Then long before the random wave packet has been dissipated by multiple scattering it will be propagating in regions with which it is *uncorrelated*, and with respect to which it may be regarded as an incident *coherent* wave. In other words, under the present circumstances, a *random* wave packet of the form (3.1) will decay according to an equation such as (3.7) for practically the whole of its lifetime.

A second random wave packet will propagate according to the same rule. From our interpretation of the scattering integral in (3.7) we see that part of the energy scattered from this second wave packet at $(\boldsymbol{x},t)$ will enter the mode $\overline{\phi}_k$ of equation (3.1) provided that the integrand of the appropriate scattering integral is non-zero at $\boldsymbol{K}=\boldsymbol{k}$. Since this energy is *uncorrelated* with that of the original wave packet, the total energy in the mode $\boldsymbol{k}$ is then obtained by simple addition of the separate energies.

Let us now attempt to formulate these ideas rather more precisely. We have been speaking of a *discrete* distribution of wave packets, but in practice energy will be distributed over a continuum of wavenumber vectors $\boldsymbol{k}$. We therefore introduce an *energy density function* $\mathscr{E}(\boldsymbol{k},\boldsymbol{x},t)$ such that $\mathscr{E}(\boldsymbol{k},\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{k}$ is equal to the wave energy per unit volume of the medium at time t and position $\boldsymbol{x}$ in the element $(\boldsymbol{k},\mathrm{d}\boldsymbol{k})$ of wavenumber space. Then according to equation (3.7) the rate at which energy is *removed* from the element $(\boldsymbol{k},\mathrm{d}\boldsymbol{k})$ of wavenumber space at $(\boldsymbol{x},t)$ per unit volume is $-I$, say, where

$$I=\frac{-2\pi\mathscr{E}(\boldsymbol{k},\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{k}}{|\partial L(\boldsymbol{k},\omega)/\partial\omega|}\int\Phi(\boldsymbol{k},\boldsymbol{K})\,\delta\{L(\boldsymbol{K},\omega)\}\,\mathrm{d}\boldsymbol{K},$$

i.e. from (2.21)

$$I=\frac{-2\pi\mathscr{E}(\boldsymbol{k},\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{k}}{|\partial L(\boldsymbol{k},\omega)/\partial\omega|}\int\Phi(\boldsymbol{k},\boldsymbol{K})\,\delta\{P(\boldsymbol{K})\,\omega(\boldsymbol{k})^2-Q(\boldsymbol{K})\}\,\mathrm{d}\boldsymbol{K}. \tag{3.8}$$

But by what has been said above, a second wave packet of energy $\mathscr{E}(\boldsymbol{K},\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{K}$ per unit volume scatters energy at a rate equal to

$$\frac{2\pi\mathscr{E}(\boldsymbol{K},\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{K}}{|\partial L(\boldsymbol{K},\omega)/\partial\omega|}\Phi(\boldsymbol{K},\boldsymbol{k})\,\delta\{P(\boldsymbol{k})\,\omega(\boldsymbol{K})^2-Q(\boldsymbol{k})\}\,\mathrm{d}\boldsymbol{k}$$

per unit volume into $(\boldsymbol{k}, \mathrm{d}\boldsymbol{k})$ at $(\boldsymbol{x}, t)$. Summing over all such wave packets $(\boldsymbol{K}, \mathrm{d}\boldsymbol{K})$ we deduce that the rate at which energy is scattered into $(\boldsymbol{k}, \mathrm{d}\boldsymbol{k})$ at $(\boldsymbol{x}, t)$ from *all* other wave modes is given by

$$J = 2\pi \mathrm{d}\boldsymbol{k} \int \frac{\mathscr{E}(\boldsymbol{K}, \boldsymbol{x}, t)}{|\partial L(\boldsymbol{K}, \omega)/\partial \omega|} \Phi(\boldsymbol{K}, \boldsymbol{k}) \delta\{P(\boldsymbol{k})\, \omega(\boldsymbol{K})^2 - Q(\boldsymbol{k})\}\, \mathrm{d}\boldsymbol{K} \tag{3.9}$$

per unit volume.

Now in equation (3.8) $\omega(\boldsymbol{k})^2 = Q(\boldsymbol{k})/P(\boldsymbol{k})$, and in (3.9) $\omega(\boldsymbol{K})^2 = Q(\boldsymbol{K})/P(\boldsymbol{K})$. Hence combining the two expressions:

$$I + J = \frac{\pi \mathrm{d}\boldsymbol{k}}{|\omega(\boldsymbol{k})|} \int [\mathscr{E}(\boldsymbol{K}, \boldsymbol{x}, t) - \mathscr{E}(\boldsymbol{k}, \boldsymbol{x}, t)]\, \Phi(\boldsymbol{k}, \boldsymbol{K}) \delta\{P(\boldsymbol{K})\, Q(\boldsymbol{k}) - P(\boldsymbol{k})\, Q(\boldsymbol{K})\}\, \mathrm{d}\boldsymbol{K}, \tag{3.10}$$

where it has been noted that $|\omega(\boldsymbol{k})| = |\omega(\boldsymbol{K})|$, i.e. frequency is conserved on scattering,

$$\Phi(\boldsymbol{k}, \boldsymbol{K}) \equiv \Phi(\boldsymbol{K}, \boldsymbol{k}),$$

and $\delta(\alpha x) = \delta(x)/|\alpha|$.

We are now in a position to write down the kinetic equation for an ensemble of random wave packets. To do this it is merely necessary to observe that by the above the following modified form of equation (3.7), describing the *net* energy exchange processes associated with a wave packet $\mathscr{E}(\boldsymbol{k}, \boldsymbol{x}, t)\, \mathrm{d}\boldsymbol{k}$, must be adopted:

$$\frac{\partial}{\partial t}[\mathscr{E}(\boldsymbol{k}, \boldsymbol{x}, t)\, \mathrm{d}\boldsymbol{k}] + \frac{\partial \omega}{\partial k_j} \frac{\partial}{\partial x_j}[\mathscr{E}(\boldsymbol{k}, \boldsymbol{x}, t)\, \mathrm{d}\boldsymbol{k}] = I + J, \tag{3.11}$$

i.e.

$$\frac{\partial \mathscr{E}}{\partial t}(\boldsymbol{k}, \boldsymbol{x}, t) + \frac{\partial \omega}{\partial k_j} \frac{\partial \mathscr{E}}{\partial x_j}(\boldsymbol{k}, \boldsymbol{x}, t) = \frac{\pi}{|\omega(\boldsymbol{k})|} \int \Phi(\boldsymbol{k}, \boldsymbol{K})\, [\mathscr{E}(\boldsymbol{K}, \boldsymbol{x}, t) - \mathscr{E}(\boldsymbol{k}, \boldsymbol{x}, t)] \times \delta\{P(\boldsymbol{K})\, Q(\boldsymbol{k}) - P(\boldsymbol{k})\, Q(\boldsymbol{K})\}\, \mathrm{d}\boldsymbol{K}. \tag{3.12}$$

This is the desired kinetic equation describing multiple scattering in the random medium. In general, it is to be solved subject to boundary conditions which specify the initial distribution of wave energy with position in space and in wavenumber space.

4. Discussion of the kinetic equation

The kinetic equation (3.12) is expected to give a valid description of multiple scattering in an arbitrary random medium provided that the correlation scale l of the random inhomogeneities is small compared with the length δ, say, associated with the decay of coherence in the wave field. Note again that frequency $\omega(\boldsymbol{k})$ (henceforth assumed positive) is conserved in the scattering process – a consequence of the inhomogeneities being time independent. Hence only a finite number of wavenumbers $|\boldsymbol{k}|$ can propagate at a given frequency ω, so that the condition $\delta \gg l$ can always be realized for sufficiently small ϵ.

It will be remarked that the spirit of the derivation of the kinetic equation given in §3 is closely allied to that embodied in the classical kinetic theory of gases (see Chapman & Cowling 1970, especially §17). In that theory the counterpart of our result is the famous equation due to Boltzmann. The chief difference is in the form of the energy exchange mechanisms. The exchange of energy amongst the various modes of the random wave field is due to the interaction of the field with the random inhomogeneities, whereas the corresponding process for gases is a *nonlinear* effect due to molecular collisions. Similar processes in random wave theory occur in the more difficult problems associated with dynamic wave–wave interactions, to which reference has

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already been made in the introduction. Our problem is much easier in that the *collision* integral on the right of (3.12) is *linear* in the energy density $\mathscr{E}(\boldsymbol{k})$.

Actually this resemblance between the kinetic equation and the classical Boltzmann equation is rather more than superficial. To illustrate this we shall obtain the analogues of a few well-known classical results. This will both illuminate the structure of the equation and emphasize the connexion with thermodynamic concepts.

Boltzmann's H-theorem

Define the wave packet 'number density' $n(\boldsymbol{k}, \boldsymbol{x}, t)$ by

$$n(\boldsymbol{k}, \boldsymbol{x}, t) = \frac{\mathscr{E}(\boldsymbol{k}, \boldsymbol{x}, t)}{\hbar\omega(\boldsymbol{k})}, \tag{4.1}$$

where $\hbar$ is Planck's constant which is introduced to make the equation dimensionless. Then the *entropy* associated with a wave packet, S^k per unit volume of medium, is given by

$$S^k = -n(\boldsymbol{k}, \boldsymbol{x}, t) \ln (n(\boldsymbol{k}, \boldsymbol{x}, t))\, \mathrm{d}\boldsymbol{k}, \tag{4.2}$$

(see, for example, Landau & Lifshitz 1959). The total entropy density is therefore given by

$$S = -\int n(\boldsymbol{k}) \ln [n(\boldsymbol{k})]\, \mathrm{d}\boldsymbol{k}. \tag{4.3}$$

Also the entropy *flux* associated with a packet is

$$\boldsymbol{S}^k = -\frac{\partial\omega}{\partial\boldsymbol{k}} n(\boldsymbol{k}, \boldsymbol{x}, t) \ln [n(\boldsymbol{k}, \boldsymbol{x}, t)]\, \mathrm{d}\boldsymbol{k}, \tag{4.4}$$

and the total flux is

$$\boldsymbol{S} = -\int \frac{\partial\omega}{\partial\boldsymbol{k}}(\boldsymbol{k})\, n(\boldsymbol{k}) \ln [n(\boldsymbol{k})]\, \mathrm{d}\boldsymbol{k}. \tag{4.5}$$

We may now form the equation of conservation of entropy:

$$\frac{\partial S}{\partial t} + \operatorname{div} \boldsymbol{S} = -\int \frac{1}{\hbar\omega(\boldsymbol{k})} [1 + \ln [n(\boldsymbol{k})]] \left[\frac{\partial\mathscr{E}}{\partial t}(\boldsymbol{k}) + \frac{\partial\omega}{\partial\boldsymbol{k}} \cdot \frac{\partial\mathscr{E}}{\partial\boldsymbol{x}}(\boldsymbol{k})\right] d\boldsymbol{k}. \tag{4.6}$$

With the use of the kinetic equation (3.12) this becomes

$$\begin{aligned}\frac{\partial S}{\partial t} + \operatorname{div} \boldsymbol{S} &= -\pi \int \frac{1}{\hbar\omega(\boldsymbol{k})^2} [1 + \ln n(\boldsymbol{k})] [\mathscr{E}(\boldsymbol{K}) - \mathscr{E}(\boldsymbol{k})]\, \Phi(\boldsymbol{k}, \boldsymbol{K})\, \delta\{P(\boldsymbol{K})\, Q(\boldsymbol{k}) - P(\boldsymbol{k})\, Q(\boldsymbol{K})\}\, \mathrm{d}\boldsymbol{K}\, \mathrm{d}\boldsymbol{k} \\ &= -\pi \int \frac{1}{\hbar\omega(\boldsymbol{K})^2} [1 + \ln n(\boldsymbol{K})] [\mathscr{E}(\boldsymbol{k}) - \mathscr{E}(\boldsymbol{K})]\, \Phi(\boldsymbol{K}, \boldsymbol{k})\, \delta\{P(\boldsymbol{k})\, Q(\boldsymbol{K}) - P(\boldsymbol{K})\, Q(\boldsymbol{k})\}\, \mathrm{d}\boldsymbol{k}\, \mathrm{d}\boldsymbol{K}.\end{aligned}$$

Adding these two expressions, and noting that for non-trivial contributions to the integrals $\omega(\boldsymbol{k}) = \omega(\boldsymbol{K})$, and also that $\Phi(\boldsymbol{k}, \boldsymbol{K}) \equiv \Phi(\boldsymbol{K}, \boldsymbol{k})$ we find that

$$\frac{\partial S}{\partial t} + \operatorname{div} \boldsymbol{S} = -\tfrac{1}{2}\pi \int \frac{1}{\omega(\boldsymbol{k})} \ln \left(\frac{n(\boldsymbol{k})}{n(\boldsymbol{K})}\right) [n(\boldsymbol{K}) - n(\boldsymbol{k})]\, \Phi(\boldsymbol{k}, \boldsymbol{K})\, \delta\{P(\boldsymbol{K})\, Q(\boldsymbol{k}) - P(\boldsymbol{k})\, Q(\boldsymbol{K})\}\, \mathrm{d}\boldsymbol{K}\, d\boldsymbol{k}. \tag{4.7}$$

Next the argument familiar in the classical kinetic theory of gases, namely, that

$$[n(\boldsymbol{K}) - n(\boldsymbol{k})] \ln \left(\frac{n(\boldsymbol{k})}{n(\boldsymbol{K})}\right) \leqslant 0 \tag{4.8}$$

leads to the conclusion that

$$\partial S/\partial t + \operatorname{div} \boldsymbol{S} \geqslant 0. \tag{4.9}$$

This is just an expression of the Second Law of Thermodynamics, and states that the total entropy of the system of wave packets cannot decrease. By associating a quantity H with $-S$ this becomes the usual form of the Boltzmann H theorem; reference to the classical analysis may be made to §4.1 of the treatise of Chapman & Cowling (1970).

Notice also that the total entropy is conserved provided only that the left-hand side of (4.8) vanishes identically, i.e. for

$$\mathscr{E}(\boldsymbol{K}, \boldsymbol{x}, t) = \mathscr{E}(\boldsymbol{k}, \boldsymbol{x}, t), \tag{4.10}$$

where $\omega(\boldsymbol{K}) = \omega(\boldsymbol{k})$. In other words, when there is *equipartition of energy* amongst all the admissible modes of each frequency. Actually the form of the dispersion function (2.21) also implies that the divergence term in (4.6) would then be identically zero, so that the distribution of energy is also independent of position and time.

The equality (4.10) implies also that

$$\mathscr{E}(\boldsymbol{k})\,\Phi(\boldsymbol{K},\boldsymbol{k})\,\delta\{P(\boldsymbol{K})\,Q(\boldsymbol{k})-P(\boldsymbol{k})\,Q(\boldsymbol{K})\}\,\mathrm{d}\boldsymbol{K}\,\mathrm{d}\boldsymbol{k} = \mathscr{E}(\boldsymbol{K})\,\Phi(\boldsymbol{k},\boldsymbol{K})\,\delta\{P(\boldsymbol{k})\,Q(\boldsymbol{K})-P(\boldsymbol{K})\,Q(\boldsymbol{k})\}\,\mathrm{d}\boldsymbol{k}\,\mathrm{d}\boldsymbol{K}, \tag{4.11}$$

which is an analytical expression of the *principle of detailed balancing*. This states that in equilibrium the wave energy scattered into the element $(\boldsymbol{k}, \mathrm{d}\boldsymbol{k})$ from the element $(\boldsymbol{K}, \mathrm{d}\boldsymbol{K})$ is *exactly* balanced by the inverse process.

Since functions of $\omega(\boldsymbol{k})$ are the only quantities conserved on scattering, it is clear that steady-state solutions of the kinetic equation must have the form $\mathscr{E} = \mathscr{E}(\omega(\boldsymbol{k}))$.

Conservation of energy

In constructing the kinetic equation (3.12) care was taken to ensure that locally there is an energy balance in the scattering process. This means that globally solutions of the equation should also satisfy the condition of conservation of energy. That this is indeed the case is established by multiplying equation (3.12) by $\mathrm{d}\boldsymbol{k}$ and integrating over all $\boldsymbol{k}$. By symmetry the double integral

$$\pi\iint \frac{[\mathscr{E}(\boldsymbol{K})-\mathscr{E}(\boldsymbol{k})]}{\omega(\boldsymbol{k})}\,\Phi(\boldsymbol{k},\boldsymbol{K})\,\delta\{P(\boldsymbol{K})\,Q(\boldsymbol{k})-P(\boldsymbol{k})\,Q(\boldsymbol{K})\}\,\mathrm{d}\boldsymbol{K}\,\mathrm{d}\boldsymbol{k}$$

vanishes identically, leaving the *macroscopic* energy conservation equation:

$$\frac{\partial}{\partial t}\int \mathscr{E}(\boldsymbol{k})\,\mathrm{d}\boldsymbol{k} + \mathrm{div}\int \frac{\partial\omega}{\partial\boldsymbol{k}}(\boldsymbol{k})\,\mathscr{E}(\boldsymbol{k})\,\mathrm{d}\boldsymbol{k} = 0, \tag{4.12}$$

in which variations in the total wave energy density $\int \mathscr{E}(\boldsymbol{k})\,\mathrm{d}\boldsymbol{k}$ are balanced by the net flux of energy represented by the divergence term. In this respect, therefore, the kinetic equation formulation of multiple scattering is seen to be superior to the lowest order approximation to the Bethe–Salpeter equation discussed in §2 which, on the contrary, does *not* appear to conserve energy.

Consider next the problem of scattering of very short waves, i.e. those waves for which a typical wavenumber $\boldsymbol{k}$ satisfies $kl \gg 1$. Provided that ϵ the parameter associated with the magnitude of the inhomogeneities of the medium, is sufficiently small, it is still possible to ensure that the dissipation scale δ is large compared with the correlation length l, so that the present theory should still be applicable. This case is significant because the scattered field is dominated by 'forward scatter', in which practically *all* of the scattered energy propagates in the same direction as the incident wave. This occurs since we are really in the limit of 'geometrical optics' in which variations in the properties of the medium are slow on a scale of wavelength. The major part of the scattered field should now be interpreted as a *phase shift* which is due to the fact that the wave is

travelling at the slowly varying *local* group velocity. Classical scattering theories, in assuming that all the scattered energy is lost from the incident wave, therefore predict the embarrassing result that the incident wave is rapidly annihilated, whereas it is actually only the initial *coherence* which is destroyed, most of the scattered energy still propagating along the geometrical optics path. Equation (3.12) shows that such forward scatter anomalies are absent from the present theory, since in the forward direction $\mathscr{E}(\boldsymbol{K}) = \mathscr{E}(\boldsymbol{k})$ and the integrand of the energy exchange integral is exactly zero. This aspect of the problem will be examined further in the next section. It suffices to suggest here that the present approach appears to have application to the theory of *sonic boom* propagation through the turbulent atmospheric boundary layer of the Earth.

Actually the case considered above is closely related to the problem of propagation through a medium whose *mean* properties vary slowly in space. In that case the kinetic equation must be modified to take account of the variation in the wavenumber $\boldsymbol{k}$ of a wave packet produced by the variations in the mean properties of the medium. The appropriate form of the equation is then

$$\frac{\partial \mathscr{E}}{\partial t}(\boldsymbol{k}) + \frac{\partial \omega}{\partial k_j}\frac{\partial \mathscr{E}}{\partial x_j}(\boldsymbol{k}) - \frac{\partial \omega}{\partial x_j}\frac{\partial \mathscr{E}}{\partial k_j}(\boldsymbol{k}) = \frac{\pi}{\omega(\boldsymbol{k})}\int [\mathscr{E}(\boldsymbol{K}) - \mathscr{E}(\boldsymbol{k})]\, \Phi(\boldsymbol{k}, \boldsymbol{K})\, \delta\{P(\boldsymbol{K})\, Q(\boldsymbol{k}) - P(\boldsymbol{k})\, Q(\boldsymbol{K})\}\, \mathrm{d}\boldsymbol{K}. \tag{4.13}$$

Finally we note the limitations to be observed in applying the kinetic equation to *bounded* regions of random inhomogeneities. We have in mind problems which, though posed in an unbounded medium, are such that only finite or semi-infinite regions contain inhomogeneities. A careful analysis reveals that the kinetic equation (3.12) is still expected to be valid in such a region except within a layer of thickness l of the boundaries. Since it is assumed that $l \ll \delta$, scattering processes are not significant in such boundary-layer zones. Alternatively: such a layer has infinitesimal width on a scale of order $1/c^2$. Note also that in free space equation (3.12) remains valid with the right-hand side set equal to zero.

In connexion with the scattering of an incident wave by a bounded region of inhomogeneities, the following remarks are in order. If the incident wave has wavenumber $\boldsymbol{k}_0$ the corresponding energy spectrum consists of a 'spike' at $\boldsymbol{k} = \boldsymbol{k}_0$. When the wave interacts with the inhomogeneities in accordance with (3.12) it excites all possible wavenumbers $\boldsymbol{k}$ satisfying $L(\boldsymbol{k}, \omega(\boldsymbol{k}_0)) = 0$. In general, the spectrum of the scattered radiation will consist of several spikes defined by this equation. When the incident wave is 'switched on' it excites these modes which proceed to radiate as a scattered field. Equilibrium is attained when the energy extracted from the incident wave by the inhomogeneities is precisely equal to that radiated into free space in the scattered field.

5. Illustrative applications of the theory

5.1

The object of the present section is to illustrate several important aspects of the kinetic theory of multiple scattering. We shall confine ourselves to simple, *non-dispersive* media where if ϕ denotes a scalar field variable, then the wave system is governed by a Lagrangian density of the form

$$\mathscr{L} = \frac{1}{2c^2}[1 + \xi(\boldsymbol{x})]\left(\frac{\partial \phi}{\partial t}\right)^2 - \frac{1}{2}\left(\frac{\partial \phi}{\partial x_i}\right)^2. \tag{5.1}$$

In terms of the general definition (2.7) this involves three differential operators Q, H, K, say, given by $Q = 1, H \equiv K = \partial/\partial x_i$.

Equation (5.1) defines a classical wave-bearing system. When only one space dimension is involved it may be regarded as describing propagation along a stretched string whose density is a random function of position:

$$\frac{\partial^2\phi}{\partial x^2}-\frac{1+\xi(x)}{c^2}\frac{\partial^2\phi}{\partial t^2}=0. \tag{5.2}$$

In three dimensions we have

$$\nabla^2\phi-\frac{1+\xi(\boldsymbol{x})}{c^2}\frac{\partial^2\phi}{\partial t^2}=0, \tag{5.3}$$

where

$$\nabla^2=\partial^2/\partial x^2+\partial^2/\partial y^2+\partial^2/\partial z^2.$$

With respect to the notation of § 2

$$\left.\begin{aligned} G&=\frac{\xi(\boldsymbol{x})}{c^2}\frac{\partial^2}{\partial t^2},\\ L&=-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial x_i^2}, \end{aligned}\right\} \tag{5.4}$$

and it is a simple matter to deduce from (2.13) and (2.25) that

$$\Phi(\boldsymbol{k},\boldsymbol{K})=(\omega/c)^4\Phi(\boldsymbol{k}-\boldsymbol{K})=k^4\Phi(\boldsymbol{k}-\boldsymbol{K}), \tag{5.5}$$

where if

$$R(\boldsymbol{x}-\boldsymbol{y})=\overline{\xi(\boldsymbol{x})\,\xi(\boldsymbol{y})},$$

then

$$\Phi(\boldsymbol{k})=\frac{1}{(2\pi)^n}\int_{-\infty}^{\infty}R(\boldsymbol{x})\exp\left[-\mathrm{i}\boldsymbol{k}.\boldsymbol{x}\right]\mathrm{d}\boldsymbol{x}, \tag{5.6}$$

and n is the number of space dimensions involved in the problem (one or three in the cases to be considered below).

It follows from equation (3.2) that the mean rate of decay of an ensemble average wave packet of the form

$$\overline{\phi}_k=a(\boldsymbol{x},t)\exp\left[\mathrm{i}\{\boldsymbol{k}.\boldsymbol{x}-ckt\}\right]+\text{c.c.}$$

(where we have used the dispersion relation $L(\boldsymbol{k},\omega)\equiv\omega^2/c^2-k^2=0$ to eliminate the frequency ω) is given by

$$\langle P_{\mathrm{M}}^{k}\rangle=-2\pi|a|^2ck^5\int\Phi(\boldsymbol{k}-\boldsymbol{K})\,\delta(k^2-K^2)\,\mathrm{d}\boldsymbol{K}. \tag{5.7}$$

Using this result and the argument of § 3 we can now write down the appropriate kinetic equation, in which we make the formal substitution

$$\mathscr{E}(\boldsymbol{k},\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{k}=\omega\frac{\partial L}{\partial\omega}(\boldsymbol{k},\omega)\,|a|^2=2k^2|a|^2, \tag{5.8}$$

namely,

$$\frac{\partial\mathscr{E}}{\partial t}(\boldsymbol{k})+\frac{c\boldsymbol{k}}{k}.\frac{\partial\mathscr{E}}{\partial\boldsymbol{x}}(\boldsymbol{k})=\pi ck^3\int\Phi(\boldsymbol{k}-\boldsymbol{K})\left[\mathscr{E}(\boldsymbol{K})-\mathscr{E}(\boldsymbol{k})\right]\delta(k^2-K^2)\,\mathrm{d}\boldsymbol{K}. \tag{5.9}$$

This is an extremely simple integro-differential equation describing the evolution of the energy field. The advantage of using this equation lies in the possibility of demonstrating in a fairly straightforward manner certain of the fundamental implications of the theory. The following discussion will be restricted to a consideration of three rather elementary problems which are, however, extremely cumbersome, if not impossible, to treat by classical scattering theories. The first two problems will deal with the case of propagation in one dimension, and then finally we shall consider a three dimensional problem concerning scattering by an inhomogeneous halfspace.

5.2. *Transmission and reflexion by a slab of random medium*

The implications of applying the kinetic equation to bounded random regions have already been referred to in §4. Consider the case of *one-dimensional* propagation in which only a slab-like region $\mathscr{R}$ of width h, say, contains random inhomogeneities (equation (5.2)). If l denotes the correlation scale of these random inhomogeneities, then there will be a significant scattered field provided that $h \gg l$.

The integral on the right of the kinetic equation (5.9) can be performed immediately in the one-dimensional case using the properties of the Dirac delta function, so that the kinetic equation assumes the especially simple form

$$\frac{\partial \mathscr{E}}{\partial t}(k) + c\,\mathrm{sgn}\,(k)\frac{\partial \mathscr{E}}{\partial x}(k) = \tfrac{1}{2}\pi c k^2 \Phi(2k)\{\mathscr{E}(-k) - \mathscr{E}(k)\}, \tag{5.10}$$

within the region $\mathscr{R}$, and

$$\frac{\partial \mathscr{E}}{\partial t}(\boldsymbol{k}) + c\,\mathrm{sgn}\,(k)\frac{\partial \mathscr{E}}{\partial x}(k) = 0 \tag{5.11}$$

outside $\mathscr{R}$.

Suppose that the region $\mathscr{R}$ extends over the range $0 < x < h$, and consider the steady state problem of a plane wave for which $k = k_0 > 0$ incident on $\mathscr{R}$ from $x < 0$. The energy density of this wave in wavenumber space may be represented by

$$\mathscr{E}_0(k) = I_0\delta(k - k_0) \quad (x < 0). \tag{5.12}$$

It is evident from the form of the scattering term on the right of (5.10) that the spectrum of the scattered field will consist of two distinct peaks at $k = \pm k_0$, corresponding to forward and back scatter at the frequency of the incident wave. Hence we may quite generally set

$$\mathscr{E}(k) = I_+(x)\delta(k - k_0) + I_-(x)\delta(k + k_0). \tag{5.13}$$

The functions $I_+(x)$, $I_-(x)$ are determined by substitution into (5.10), in which case it is readily deduced that they satisfy

$$\left.\begin{aligned}\frac{\partial I_+}{\partial x} &= \frac{1}{\Lambda}(I_- - I_+),\\ \frac{\partial I_-}{\partial x} &= \frac{1}{\Lambda}(I_- - I_+),\end{aligned}\right\} \tag{5.14}$$

in $0 < x < h$, and they are *constant* elsewhere. In (5.14)

$$\Lambda = 1/(\tfrac{1}{2}\pi k_0^2 \Phi(2k_0)), \tag{5.15}$$

and is the dissipation scale associated with the intensity of the *back scatter* experienced by a wave. It is generally much larger than the relaxation scale associated with the decay of *coherence*.

A unique solution of the system (5.14) is determined by the imposition of two boundary conditions. These are furnished by the radiation condition which requires that all *scattered* waves radiate *away* from the scattering region $\mathscr{R}$. Of course account must be taken of the input from the incident wave, so that the appropriate conditions are just

$$\left.\begin{aligned}I_+ &= I_0 \quad \text{at} \quad x = 0,\\ I_- &= 0 \quad \text{at} \quad x = h,\end{aligned}\right\} \tag{5.16}$$

from which it follows that

$$\left.\begin{aligned} I_+ &= I_0\left(1-\frac{x}{h+\Delta}\right)\\ I_- &= I_0\left(\frac{h-x}{h+\Delta}\right)\end{aligned}\right\} \quad (0<x<h). \tag{5.17}$$

This solution illustrates an important feature common to all steady multiple scattering problems. In the first instance $I_+ - I_- = I_0\Delta/(h+\Delta)$, and when $h \gg \Delta$ the steady state distribution of energy within $\mathscr{R}$ is essentially one of local equipartition between the two possible wave modes. The balance is not exact, however, and the residual difference in intensity gives a transmitted field in $x > h$ precisely equal to $I_0\Delta/(h+\Delta)$. Thus for $h \gg \Delta$ very little energy succeeds in penetrating into $x > h$. The reflected intensity is then very nearly equal to that of the incident wave, as would be expected by energy conservation arguments.

In terms of a transmission coefficient T and a reflexion coefficient R we have actually shown that

$$\langle|T^2|\rangle = \frac{\Delta}{h+\Delta}; \quad \langle|R^2|\rangle = \frac{h}{h+\Delta}. \tag{5.18}$$

In the theory of radiative transfer (Sobolev 1963) $\langle|R|^2\rangle$ is usually referred to as the *albedo* of the inhomogeneities. The interesting point to note about these expressions is that, unlike a naive scattering theory approach, the transmitted field does not decay exponentially as the width h of $\mathscr{R}$ increases. Such theories neglect the contribution of 'forward scatter'; our result indicates that its cumulative effect produces a transmitted field which is only algebraically small.

Comparison may also be made with the recent work of Morrison, Papanicolaou & Keller (1971) concerning the calculation of the mean power transmitted through a random slab. That paper contains a highly sophisticated analysis based on the idea of treating the density inhomogeneities as a Markov process, following a suggestion due to Frisch (1968). Their analytical results are qualitatively similar to those embodied in (5.18).

5.3. *The scattering theory of geometrical optics*

It was pointed out in §4 that the kinetic equation is expected to remain valid in the limit of short wavelength provided that $\delta \ll l$, where δ is the dissipation scale associated with the decay of coherence and l is the correlation length of the inhomogeneities. In other words, in that limit the results of geometrical optics should be recovered. The reason for the failure of conventional scattering theories in this case is that most, but not all, of the 'scattered' energy radiates in the *same* direction and at the *same* velocity as the incident wave, and should not really be counted as an energy loss.

On the other hand, the approximation of geometrical optics (or W.K.B. approximation) would not be expected to be efficient for very large distances and times. Indeed that theory assumes that a wave packet propagates through a slowly varying medium without significant attenuation due to scattering. Such scattering certainly proceeds at an extremely slow rate, but its effect is cumulative and ultimately dominates the evolution of the wave field. This may be illustrated by means of a simple one-dimensional model problem which contains all the basic features of the general case. Actually because the scattering is either 'forward' or 'back' the interpretation is not limited to high wavenumbers in the case of one-dimensional problems. The interpretation in two or more dimensions *is* restricted to large wavenumbers, however, since *forward scatter* is only an important issue in that limit.

Suppose that at time $t = 0$ a wave packet localized about the origin $x = 0$ is created and proceeds to propagate according to equation (5.2) in the positive direction of the x-axis. The energy density in the (k, x) *phase space* may be represented initially by

$$\mathscr{E}(k, x, 0) = I_0 \delta(x)\delta(k - k_0) \quad (t = 0). \tag{5.19}$$

The presence of the function $\delta(x)$ merely implies that the dimensions of the wave packet are small on a scale of order $1/\epsilon^2$.

Again for $t > 0$ it is evident that the spectrum of the wave field has the form given in (5.13), except that the functions I_+, I_- depend also on the time, and satisfy

$$\left.\begin{aligned} \frac{\partial I_+}{\partial t} + c\frac{\partial I_+}{\partial x} &= \frac{c}{\Delta}(I_- - I_+), \\ \frac{\partial I_-}{\partial t} - c\frac{\partial I_-}{\partial x} &= -\frac{c}{\Delta}(I_- - I_+), \end{aligned}\right\} \tag{5.20}$$

with $I_- = 0$, $I_+ = I_0\delta(x)$ at $t = 0$.

Setting $I_\pm = J_\pm \mathrm{e}^{-ct/\Delta}$ it is an easy matter to show that

$$\frac{\partial^2 J_\pm}{\partial t^2} - c^2\frac{\partial^2 J_\pm}{\partial x^2} - \frac{c^2}{\Delta^2}J_\pm = 0, \tag{5.21}$$

which is the well-known *Telegraphist's equation* (Webster 1955). By means of equations (5.20) and the initial conditions we deduce that the appropriate initial conditions for J_-, say, are

$$J_- = 0; \quad \frac{\partial J_-}{\partial t} = \frac{c}{\Delta} I_0 \delta(x) \quad \text{at} \quad t = 0.$$

Hence for $t > 0$,

$$J_- = \frac{I_0}{2\Delta} H(ct - |x|)\, \boldsymbol{I}_0\left(\frac{\sqrt{(c^2t^2 - x^2)}}{\Delta}\right),$$

i.e.

$$I_- = \frac{I_0}{2\Delta} H(ct - |x|)\, \boldsymbol{I}_0\left(\frac{\sqrt{(c^2t^2 - x^2)}}{\Delta}\right) \mathrm{e}^{-ct/\Delta} \tag{5.22}$$

(Webster 1955, p. 253). Similarly, we can show that

$$I_+ = I_0\delta(x - ct)\, \mathrm{e}^{-ct/\Delta} + \frac{I_0}{2\Delta} H(ct - |x|) \sqrt{\frac{t + x/c}{t - x/c}}\, \boldsymbol{I}_1\left(\frac{\sqrt{(c^2t^2 - x^2)}}{\Delta}\right) \mathrm{e}^{-ct/\Delta}. \tag{5.23}$$

In these results $\boldsymbol{I}_0$, $\boldsymbol{I}_1$ are modified Bessel functions of the first kind, and H is the Heaviside unit function.

These solutions illustrate in a rather precise fashion the effects mentioned above. First recall that Δ is a dissipation scale associated with the strength of back scatter (equation (5.15)). When the wave packet has travelled distances ct which are small compared to Δ the solutions indicate that the field consists essentially of an unmodified wave packet together with a weak 'tail' in its wake. Actually this tail is diffusing into $x < 0$ at the same speed of propagation of the packet. It consists of waves (5.22) propagating in the negative x-direction, initially generated by back scatter, together with waves moving in the *same* direction as the incident packet (second term of (5.23)) generated by multiple scattering.

When the packet has travelled several scales Δ, the exponential factor in the solution shows that the energy content of the *wave packet* becomes negligibly small. The Bessel function terms in

(5.22) and (5.23), however, do not decay so rapidly, and indeed now dominate the field. In fact as ct/x becomes large

$$I_+ \simeq I_- \simeq \frac{e^{-x^2/2\Delta ct}}{2\sqrt{(2\pi\Delta ct)}}, \tag{5.24}$$

indicating that the ultimate distribution of energy is one of local equipartition between both possible wave modes. The field now consists of a 'cloud' of energy gradually diffusing to infinity.

We therefore conclude that the initial concentrated energy distribution has been dispersed uniformly, and that the wave packet may be assigned a 'half-life' $\tau = \Delta/c$. The latter may be regarded as the time over which geometrical optics would be expected to give an efficient representation of the wave field. It is clear from the definition of Δ given in equation (5.15) that for sufficiently large wavenumber k_0, Δ can be very large, since $k_0^2\Phi(2k_0) \to 0$ as $k_0 \to \infty$.

5.4. *Scattering by an inhomogeneous half-space*

Let us now consider a more complicated problem involving a three-dimensional random medium specified by equation (5.3). It will be assumed that the random inhomogeneities occupy only the half-space $x > 0$. Then a plane wave incident on this region will give rise to a scattered field in the free space $x < 0$. We shall use the kinetic theory to determine the angular distribution of this scattered field, and examine also the general properties of the field established within the random half-space.

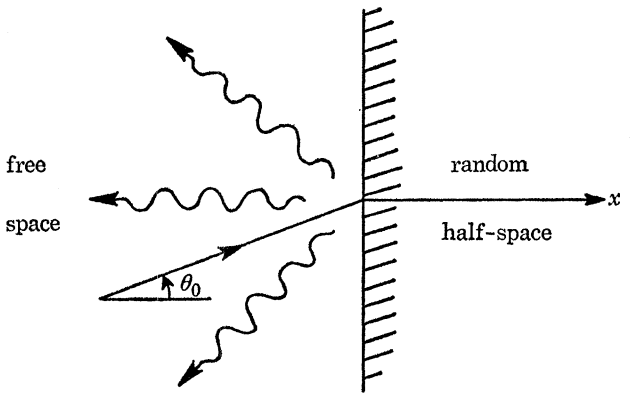

FIGURE 1. A plane wave is incident on the inhomogeneous half-space at an angle of incidence θ_0. The wavy lines represent the field scattered back into free space.

Now for a problem that is steady in time the kinetic equation (5.9) assumes the form

$$\hat{\boldsymbol{k}}.\frac{\partial \mathscr{E}}{\partial \boldsymbol{x}}(\boldsymbol{k}) = \pi k^3 \int \Phi(\boldsymbol{k}-\boldsymbol{K})\,[\mathscr{E}(\boldsymbol{K}) - \mathscr{E}(\boldsymbol{k})]\,\delta(k^2-K^2)\,\mathrm{d}\boldsymbol{K}, \tag{5.25}$$

in $x > 0$, where $\hat{\boldsymbol{k}} = \boldsymbol{k}/k$. In the case of an incident plane wave the spatial dependence of $\mathscr{E}(\boldsymbol{k})$ must be with respect to x alone, so that if $\boldsymbol{k}$ makes an angle θ with the positive direction of the x-axis we have:

$$\cos\theta \frac{\partial \mathscr{E}}{\partial x}(\boldsymbol{k}) = \pi k^3 \int \Phi(\boldsymbol{k}-\boldsymbol{K})\,[\mathscr{E}(\boldsymbol{K}) - \mathscr{E}(\boldsymbol{k})]\,\delta(k^2-K^2)\,\mathrm{d}\boldsymbol{K}, \tag{5.26}$$

in $x > 0$.

The details of the solution of this integro-differential equation depend on the form of the spectrum $\Phi(\boldsymbol{k})$ of the random inhomogeneities, but the analysis can be simplified if it is assumed that the wavelength of the incident wave (and hence also of the scattered field) greatly exceeds

the correlation length of the inhomogeneities. This means that in (5.26) $\Phi(\boldsymbol{k}-\boldsymbol{K})$ may be replaced by $\Phi(\boldsymbol{0})$. (For a discussion of the opposite limit, involving high-frequency scattering, see Howe 1972*b*.)

Suppose that the plane wave incident on the half-space from $x < 0$ has the form

$$\exp\{\mathrm{i}(\boldsymbol{k}_0.\,\boldsymbol{x}-ck_0t)\}.$$

Then it may be assumed that its energy spectrum with respect to a suitable system of units is given by

$$\mathscr{E}_0(\boldsymbol{k}) = \delta(\boldsymbol{k}-\boldsymbol{k}_0) \quad \text{in} \quad x < 0. \tag{5.27}$$

Choose a system of polar coordinates (k, θ, φ) in wavenumber space, in which θ is measured from the positive direction of the x-axis, and such that $\boldsymbol{k}_0 = (k_0, \theta_0, 0)$ with $0 \leqslant \theta_0 < \frac{1}{2}\pi$. Then alternatively we have

$$\mathscr{E}_0(\boldsymbol{k}) = \delta(\theta-\theta_0)\,\delta(\varphi)\,\delta(k-k_0)/(k^2\sin\theta)$$

or, if $\mu = \cos\theta$, $\mu_0 = \cos\theta_0$, equivalently,

$$\mathscr{E}_0(\boldsymbol{k}) = I_0\delta(\mu-\mu_0)\,\delta(\varphi)\,\delta(k-k_0), \tag{5.28}$$

where I_0 is a constant. Thus the *directivity*, $I_0(\mu, \varphi)$, of the incident wave is defined by

$$I_0(\mu, \varphi) = I_0\delta(\mu-\mu_0)\,\delta(\varphi). \tag{5.29}$$

The scattered field has the same frequency ck_0 as the incident wave, and we may therefore denote the total energy density of the wave field in the random half-space by

$$\mathscr{E}(\boldsymbol{k}) = I(\mu, \varphi, x)\,\delta(k-k_0), \tag{5.30}$$

where $I(\mu, \varphi, x)$ is the directivity of the field. Substituting this expression into the kinetic equation (5.26), and using the approximation $\Phi(\boldsymbol{k}-\boldsymbol{K}) = \Phi(\boldsymbol{0})$, then gives

$$\mu\frac{\partial I}{\partial x}(\mu, \varphi, x) + \frac{1}{\Lambda}I(\mu, \varphi, x) = \frac{1}{4\pi\Lambda}\int_{-1}^{1}\mathrm{d}\bar{\mu}\int_0^{2\pi} I(\bar{\mu}, \bar{\varphi}, x)\,\mathrm{d}\bar{\varphi}, \tag{5.31}$$

where Λ is the relaxation length associated with the decay of coherence of a plane wave, namely

$$\Lambda = 1/(2\pi^2k_0^4\Phi(\boldsymbol{0})). \tag{5.32}$$

To determine the properties of the scattered field we transform equation (5.31) into the Milne *diffusion* equation of radiative transfer theory (Tait 1964). Set

$$\rho(x) = \int_{-1}^{1}\mathrm{d}\bar{\mu}\int_0^{2\pi} I(\bar{\mu}, \bar{\varphi}, x)\,\mathrm{d}\bar{\varphi}, \tag{5.33}$$

which is proportional to the total wave energy per unit volume of medium. Then (5.31) becomes

$$\mu\frac{\partial I}{\partial x}(\mu, \varphi, x) + \frac{1}{\Lambda}I(\mu, \varphi, x) = \frac{1}{4\pi\Lambda}\rho(x), \tag{5.34}$$

which may be integrated to give

$$I(\mu, \varphi, x) = A(\mu, \varphi)\,\mathrm{e}^{-x/\Lambda\mu} + \frac{1}{4\pi\Lambda\mu}\int_0^x \rho(\xi)\,\mathrm{e}^{(\xi-x)/\Lambda\mu}\,\mathrm{d}\xi, \quad (x > 0), \tag{5.35}$$

where $A(\mu, \varphi)$ is an arbitrary function of μ, φ. This function can be determined by applying the appropriate radiation condition.

Consider first the case $\mu = \cos\theta > 0$, i.e., where $I(\mu, \varphi, x)$ is the directivity of waves propagating in the positive direction of the x-axis. Clearly the condition that scattered waves must radiate *away* from the scattering region implies there can be no contribution to $I(\mu, \varphi, x)$ from the scattered field at $x = 0$; at this interface the only wave propagating to the right is the incident wave. In other words

$$I(\mu, \varphi, 0) = I_0 \delta(\mu - \mu_0)\,\delta(\varphi) \quad (\mu > 0). \tag{5.36}$$

Hence for $\mu > 0$,

$$A(\mu, \varphi) = I_0 \delta(\mu - \mu_0)\,\delta(\varphi). \tag{5.37}$$

Next consider waves propagating in the negative x-direction, for which $\mu = \cos\theta < 0$. We can obtain the appropriate form of the radiation condition here by noting that, if the scattering region occupied only a *finite* length $0 < x < h$, say, of the x-axis, then no radiation could enter that region from $x > h$, i.e. $I(\mu, \varphi, h) = 0$ $(\mu < 0)$. Using this in (5.35), and allowing $h \to \infty$, then gives

$$A(\mu, \varphi) = \frac{-1}{4\pi\Delta\mu} \int_0^\infty \rho(\xi)\, \mathrm{e}^{\xi/\Delta\mu}\, \mathrm{d}\xi \tag{5.38}$$

for $\mu < 0$.

Thus equation (5.35) now becomes

$$\left.\begin{aligned} I(\mu, \varphi, x) &= I_0 \delta(\mu - \mu_0)\,\delta(\varphi)\, \mathrm{e}^{-x/\Delta\mu_0} + \frac{1}{4\pi\Delta\mu} \int_0^x \rho(\xi)\, \mathrm{e}^{(\xi - x)/\Delta\mu}\, \mathrm{d}\xi \quad (\mu > 0), \\ I(\mu, \varphi, x) &= -\frac{1}{4\pi\Delta\mu} \int_x^\infty \rho(\xi)\, \mathrm{e}^{(\xi - x)/\Delta\mu}\, \mathrm{d}\xi \quad (\mu < 0), \end{aligned}\right\} \tag{5.39a, b}$$

for $x > 0$.

This system can be transformed into an integral equation for the total energy density $\rho(x)$ by integrating both equations with respect to φ over $(0, 2\pi)$, and then integrating (5.39a) with respect to μ over $(0, 1)$, and (5.39b) over $\mu = -1, 0$, and adding to give

$$\rho(x) = I_0 \mathrm{e}^{-x/\Delta\mu_0} + \int_0^\infty \rho(\xi)\, V(x - \xi)\, \mathrm{d}\xi \quad (x > 0), \tag{5.40}$$

where the *kernel* $V(x)$ is given by

$$V(x) = \frac{1}{2\Delta} \int_0^1 \mathrm{e}^{-|x|/\Delta\mu} \frac{\mathrm{d}\mu}{\mu}. \tag{5.41}$$

Equation (5.40) is the inhomogeneous Milne equation, and may be solved by the method of Wiener–Hopf (Noble 1958).

Let us first give the following definitions which are familiar in the theory of the Wiener–Hopf technique. For a function $f(x)$ define the half-range functions $f_+(x), f_-(x)$ by

$$f_+(x) = H(x) f(x), \quad f_-(x) = H(-x) f(x), \tag{5.42}$$

where $H(x)$ is the Heaviside unit function.

Suppose that $f(x)$ decays exponentially as $|x| \to \infty$, then the *half-range* Fourier transforms

$$\left.\begin{aligned} \tilde{f}_+(\kappa) &= \frac{1}{2\pi} \int_0^\infty f(x)\, \mathrm{e}^{\mathrm{i}\kappa x}\, \mathrm{d}x = \frac{1}{2\pi} \int_{-\infty}^\infty f_+(x)\, \mathrm{e}^{\mathrm{i}\kappa x}\, \mathrm{d}x, \\ \tilde{f}_-(\kappa) &= \frac{1}{2\pi} \int_{-\infty}^0 f(x)\, \mathrm{e}^{\mathrm{i}\kappa x}\, \mathrm{d}x = \frac{1}{2\pi} \int_{-\infty}^\infty f_-(x)\, \mathrm{e}^{\mathrm{i}\kappa x}\, \mathrm{d}x, \end{aligned}\right\} \tag{5.43}$$

exist, and in particular $\tilde{f}_+(\kappa)$ is a regular function of κ for $\mathrm{Im}\,(\kappa) > -\tau_1$, for some $\tau_1 > 0$, and $\tilde{f}_-(\kappa)$ is regular in $\mathrm{Im}\,(\kappa) < \tau_2$, for some $\tau_2 > 0$. Also the following inversion formulae hold:

$$f_\pm(x) = \int_{-\infty}^\infty \tilde{f}_\pm(\kappa)\, \mathrm{e}^{-\mathrm{i}\kappa x}\, \mathrm{d}\kappa \tag{5.44}$$

(Noble 1958).

Now consider equation (5.40) which is valid only for $x > 0$. Set $\rho_+(x) = H(x)\,\rho(x)$, and *define* $\rho_-(x)$ by

$$\rho_-(x) = H(-x)\int_0^\infty \rho(\xi)\,V(x-\xi)\,\mathrm{d}\xi. \tag{5.45}$$

Then equation (5.40) can be expressed in the equivalent form

$$\rho_+(x)+\rho_-(x) = H(x)\,I_0\,\mathrm{e}^{-x/\Delta\mu_0}+\int_{-\infty}^\infty \rho_+(\xi)\,V(x-\xi)\,\mathrm{d}\xi. \tag{5.46}$$

Take the Fourier transform of this equation:

$$\tilde\rho_+(\kappa)\,\{1-2\pi\tilde V(\kappa)\}+\tilde\rho_-(\kappa) = \frac{\mathrm{i}I_0\Delta\mu_0}{2\pi[\kappa\Delta\mu_0+\mathrm{i}]}. \tag{5.47}$$

Here we have used the convolution theorem (Noble 1958) to evaluate the transform of the integral term on the right of (5.46), namely, $2\pi\tilde\rho_+(\kappa)\,\tilde V(\kappa)$, where $\tilde V(\kappa)$ is the Fourier transform of $V(x)$, i.e. from (5.41)

$$\tilde V(\kappa) = \frac{1}{2\pi}\frac{\arctan(\Delta\kappa)}{\Delta\kappa}. \tag{5.48}$$

Now if the energy density $\rho_+(x)$ remains bounded as $x\to+\infty$, then its half-range transform $\tilde\rho_+(\kappa)$ is regular in the upper half of the complex κ-plane. Also it is readily verified from the integral formula (5.41) that $V(x)\sim\Delta\,\mathrm{e}^{-|x|/\Delta}/x$ as $|x|\to\infty$, and therefore by (5.45) that $\rho_-(x)$ is exponentially small at $x=-\infty$. Hence $\rho_-(\kappa)$ is regular in the half plane $\mathrm{Im}\,(\kappa)<\tau$, say, where $\tau>0$. Since $\tilde V(\kappa)$ is regular in $|\mathrm{Im}\,(\kappa)|<1/\Delta$, it follows that the Wiener–Hopf functional equation (5.47) is a relation between functions which are jointly regular in the strip $0<\mathrm{Im}\,(\kappa)<\sigma$, where $\sigma=\min\{\tau,1/\Delta\}$.

Following the usual Wiener–Hopf procedure we now express $1-2\pi\tilde V(\kappa)$ in the form $L_+(\kappa)/L_-(\kappa)$, where $L_+(\kappa)$ is regular in $\mathrm{Im}\,(\kappa)>0$, and $L_-(\kappa)$ is regular in $\mathrm{Im}\,(\kappa)<\sigma$. Then (5.47) can be expressed in the form

$$\tilde\rho_+(\kappa)\,L_+(\kappa)-\frac{\mathrm{i}I_0\Delta\mu_0L_-(-\mathrm{i}/\Delta\mu_0)}{2\pi[\kappa\Delta\mu_0+\mathrm{i}]} = -\tilde\rho_-(\kappa)\,L_-(\kappa)+\frac{\mathrm{i}I_0\Delta\mu_0[L_-(\kappa)-L_-(-\mathrm{i}/\Delta\mu_0)]}{2\pi[\kappa\Delta\mu_0+\mathrm{i}]}. \tag{5.49}$$

This equation is valid in the common strip of regularity, and the left-hand side is, in particular, regular in $\mathrm{Im}\,(\kappa)>0$ and the right-hand side is regular in $\mathrm{Im}(\kappa)<\sigma$. Therefore by the uniqueness of analytic continuation, the equation defines a unique regular function in the whole of the complex κ-plane.

By means of Cauchy's integral theorem we can deduce in the usual way that

$$L_+(\kappa) = \frac{\Delta^2\kappa^2}{\Delta\kappa+\mathrm{i}}\,\mathrm{e}^{\psi(\kappa\Delta)}, \tag{5.50}$$

where

$$\psi(\kappa\Delta) = \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^\infty \frac{\ln\left\{\left(\frac{1+\xi^2}{\xi^2}\right)\left(1-\frac{\arctan\xi}{\xi}\right)\right\}\mathrm{d}\xi}{\xi-\kappa\Delta}, \tag{5.51}$$

the path of integration passing below the singularity at $\xi=\kappa\Delta$. Now $\psi(\kappa\Delta)$ is regular and bounded in the upper half plane, so that by (5.50) $L_+(\kappa)\sim O(\kappa)$ when κ is large. Also the boundedness of $\rho_+(x)$ implies that $\tilde\rho_+(\kappa)$ is at least $O(1/\kappa)$ as $|\kappa|\to\infty$ in the upper half plane. Hence in this limit

the left-hand side of (5.49) is bounded, and the usual argument involving Liouville's theorem (Noble 1958) then shows that

$$\tilde{\rho}_+(\kappa) = \frac{\mathrm{i}I_0 \Delta\mu_0 L_-(-\mathrm{i}/\Delta\mu_0)}{2\pi[\kappa\Delta\mu_0 + \mathrm{i}]\, L_+(\kappa)} + \frac{M}{L_+(\kappa)} \tag{5.52}$$

where M is constant.

The constant M must be chosen to ensure that $\rho_+(x)$ is bounded as $x \to +\infty$. Now this asymptotic behaviour is determined by the nature of the singularities of $\tilde{\rho}_+(\kappa)$ on the real axis. It is clear from (5.50) that $1/L_+(\kappa)$ has a double pole at $\kappa = 0$, and this is the only singularity of $\tilde{\rho}_+(\kappa)$ on the real axis. But that implies that $\rho_+(x) \sim O(x)$ as $x \to +\infty$ (Lighthill 1958). To eliminate the double pole set

$$M = -\frac{I_0}{2\pi} \Delta\mu_0 L_-(-\mathrm{i}/\Delta\mu_0), \tag{5.53}$$

in which case

$$\tilde{\rho}_+(\kappa) = -\frac{I_0 \kappa (\Delta\mu_0)^2 L_-(-\mathrm{i}/\Delta\mu_0)}{2\pi[\kappa\Delta\mu_0 + \mathrm{i}]\, L_+(\kappa)}, \tag{5.54}$$

i.e. by (5.50)

$$\tilde{\rho}_+(\kappa) = -\frac{I_0 \mu_0^2 L_-(-\mathrm{i}/\Delta\mu_0)\,[\Delta\kappa + \mathrm{i}]}{2\pi\kappa[\kappa\Delta\mu_0 + \mathrm{i}]}\, \mathrm{e}^{-\psi(\kappa\Delta)}. \tag{5.55}$$

Since the only remaining singularity on the real axis of $\tilde{\rho}_+(\kappa)$ is now a simple pole at the origin, it follows that $\rho_+(x)$ tends to a *constant* value as $x \to +\infty$.

To work out the precise details of the total energy distribution it is necessary to invert the appropriate Fourier integral involving $\tilde{\rho}_+(\kappa)$ as in (5.44). However, it is possible to derive all essential information regarding the energy distribution from the spectral representation (5.55).

First let us examine the *directivity* $I(\mu, \varphi, x)$ of the field as $x \to +\infty$. To do this return to equation (5.34) and take the positive half-range transform by multiplying by $\mathrm{e}^{\mathrm{i}\kappa x}/2\pi$ and integrating over $x = 0, \infty$. From this we have

$$\tilde{I}_+(\mu, \varphi, \kappa) = \frac{\Delta\mu I(\mu, \varphi, 0) + (1/4\pi)\,\tilde{\rho}_+(\kappa)}{[1 - \mathrm{i}\Delta\kappa\mu]}. \tag{5.56}$$

The behaviour of the directivity as $x \to +\infty$ is determined by the nature of the 'worst' singularity of the half-range transform $\tilde{I}_+(\mu, \varphi, \kappa)$ on the real axis (Lighthill 1958). Since this is just the simple pole at $\kappa = 0$ associated with $\tilde{\rho}_+(\kappa)$, it follows readily that $I(\mu, \varphi, x)$ tends to a constant value *independent* of μ and φ as $x \to +\infty$. Thus for sufficiently large x the energy density $\rho(x)$ becomes constant and the directivity of the field uniform, i.e. deep inside the random half-space the wave energy is distributed evenly amongst all possible modes (equipartition of energy), and therefore the *net flux of energy* in all directions, and in particular in the x-direction, vanishes identically. This is significant because when (5.34) is integrated over all μ and φ we obtain

$$\frac{\partial}{\partial x} \int_{-1}^{1} \mathrm{d}\mu \int_0^{2\pi} \mu I(\mu, \varphi, x)\, \mathrm{d}\varphi = 0,$$

i.e.

$$\int_{-1}^{1} \mathrm{d}\mu \int_0^{2\pi} \mu I(\mu, \varphi, x)\, \mathrm{d}\phi = \text{constant},$$

and since as $x \to +\infty$ this integral vanishes, it must vanish also at $x = +0$. This means that in the steady state *all* the incident energy is scattered back into free space.

We can determine the directivity of the 'reflected' radiation by noting that it is equal to $I(\mu, \varphi, x)$ $(\mu < 0)$ at $x = 0$, i.e. by (5.39*b*),

$$I(\mu, \varphi, 0) = -\frac{1}{4\pi\Delta\mu}\int_0^\infty \rho(\xi)\, e^{\xi/\Delta\mu}\, d\xi. \tag{5.57}$$

Now in the range $0 < x < \infty$, $\rho(x) \equiv \rho_+(x)$ can be expressed in terms of the half-range transform $\tilde{\rho}_+(\kappa)$ by means of the inversion formula (5.44). Making this substitution in (5.57) and performing the integrations yields

$$I(\mu, \varphi, 0) = \frac{1}{2\Delta|\mu|}\tilde{\rho}_+\left(\frac{i}{\Delta|\mu|}\right) \quad (\mu < 0), \tag{5.58}$$

i.e. from (5.55)

$$I(\mu, \varphi, 0) = \frac{N(\mu_0)\,[1+|\mu|]}{[\mu_0+|\mu|]}\, e^{-\psi(i/|\mu|)}, \tag{5.59}$$

where $N(\mu_0)$ is a function of the angle of incidence $\theta_0 = \arccos(\mu_0)$ alone.

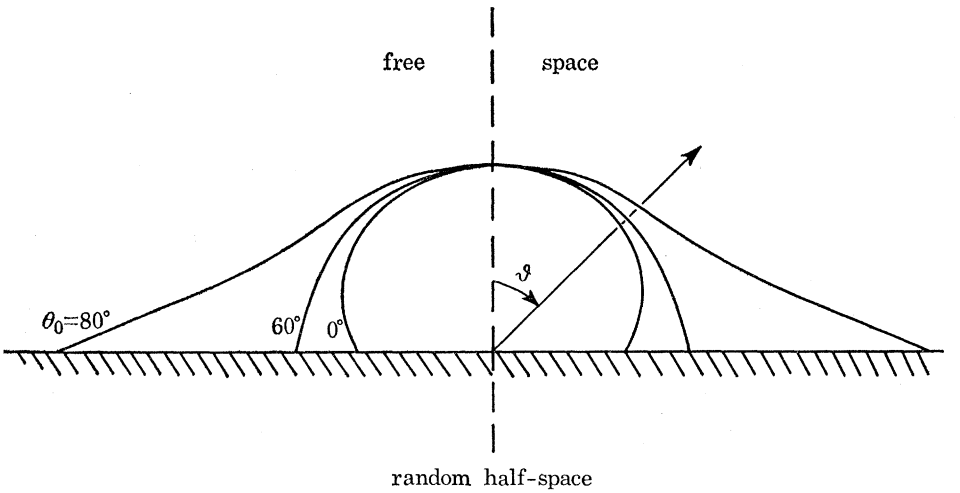

FIGURE 2. Polar plots of the normalized directivity $\mathscr{I}(\vartheta)$ of the field scattered back into free space. Three cases are shown for angles of incidence $\theta_0 = 0°, 60°, 80°$.

Let ϑ denote the angle of reflexion then, since the directivity of the reflected field is obviously *independent of the azimuthal angle* φ, we may express the directivity in the form

$$I(\vartheta) = \frac{N(\mu_0)[1+\cos\vartheta]}{\cos\theta_0+\cos\vartheta}\, e^{-\psi(\vartheta)}. \tag{5.60}$$

This can be normalized to unity in the normal direction, $\vartheta = 0$, by writing $\mathscr{I}(\vartheta) = I(\vartheta)/I(0)$, in which case we have

$$\mathscr{I}(\vartheta) = \frac{(1+\cos\theta_0)\,(1+\cos\vartheta)}{2(\cos\theta_0+\cos\vartheta)}\, e^{\psi(0)-\psi(\vartheta)}. \tag{5.61}$$

Using (5.51) the integral for $\psi(\vartheta)$ can be written as

$$\psi(\vartheta) = \frac{\cos\vartheta}{\pi}\int_0^{\frac{1}{2}\pi} \frac{\ln\left\{\frac{1}{\sin^2\varphi}\left(1-\frac{\varphi}{\tan\varphi}\right)\right\} d\varphi}{\cos^2\varphi+\sin^2\varphi\cos^2\vartheta}, \tag{5.62}$$

which is readily evaluated by straightforward numerical integration.

In figure 2 the normalized directivity $\mathscr{I}(\vartheta)$ is plotted for angles of incidence $\theta_0 = 0°, 60°, 80°$.

For ranges of θ_0 between 0°, 60° the directivity curves lie between the corresponding curves illustrated in the figure. They show no marked preferential scattering except perhaps at smaller values of θ_0 when marginally more of the scattered field radiates in the normal direction. For angles of incidence in excess of 60°, however, there is an increasing tendency for the greater part of the scattered field to be radiated in directions close to the plane of the interface. In all cases the concept of a *specularly reflected wave* (which would dominate an analysis based on the ensemble average field (see Howe 1971*b*)) is quite meaningless.

The tendency for the scattered field to be confined to directions close to the plane of the interface with increasing θ_0 may initially be rather surprising, but may be shown to be an eminently reasonable prediction by means of the following simple argument.

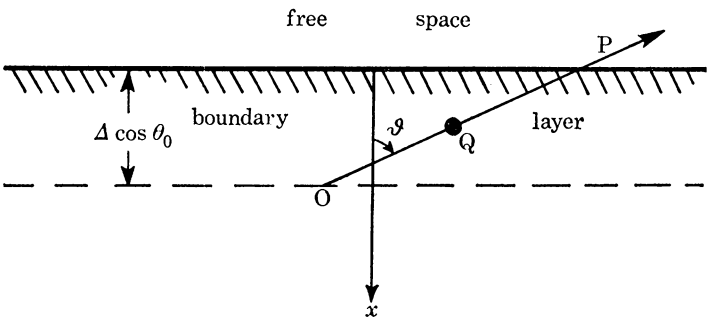

FIGURE 3. The intensity of the scattered field emerging at P in a direction making an angle ϑ with the normal to the interface depends on the length of the ray OP which lies within the 'boundary layer'.

From the foregoing discussion of the energy density we have seen that for sufficiently large x the energy distribution becomes uniform in direction, with no net transfer of energy. From the form of the half-range transform it is not difficult to see that this uniform state is established essentially for $x > \Delta\mu_0 = \Delta \cos \theta_0$, i.e. it is over this distance that information regarding the nature of the incident field is destroyed. Indeed according to (5.39*a*) the effect of the incident field would be expected to decay like $\exp(-x/\Delta \cos \theta_0)$, the second term on the right of that equation being initially small, but rising steadily to an essentially uniform value when $x > \Delta \cos \theta_0$. Since there is no net transfer of energy within this asymptotic region we may regard the scattering process responsible for the generation of the field radiated into free space as occurring within the relatively thin *boundary layer* of width $\Delta \cos \theta_0$ in which the coherence of the incident wave is destroyed. The width of this layer evidently decreases as the angle of incidence θ_0 aproaches 90°.

Suppose that an observer examines the radiation at a point P on the interface as shown in figure 3. Consider the intensity of that radiation at P in the direction of the line OP, say, which makes an angle ϑ with the normal to the interface. This intensity at P may be regarded as generated by scattering out of the coherent field at points Q on the line OP. If QP = s, then the intensity of the coherent field at Q, and therefore also of the radiation scattered at Q, is proportional to $\exp(-s \cos \vartheta/\Delta \cos \theta_0)$.

But in propagating along QP this radiation is itself attenuated by scattering by an amount proportional to $\exp(-s/\Delta)$. Hence summing over all points Q on OP, we deduce that the total scattered field emerging in the direction OP at P is, in the first approximation, proportional to

$$\int_0^\infty \exp[-s(\cos \vartheta + \cos \theta_0)/\Delta \cos \theta_0]\, \mathrm{d}s = \frac{\Delta \cos \theta_0}{\cos \vartheta + \cos \theta_0}. \tag{5.63}$$

When this is normalized to unity it gives a directivity.

$$\mathscr{I}(\vartheta) = \frac{1+\cos\theta_0}{\cos\vartheta+\cos\theta_0}. \tag{5.64}$$

This is similar to the exact formula (5.61), and illustrates the important point that when $\vartheta \sim 90°$, $\mathscr{I}(\vartheta) \sim \sec\theta_0$, which is large when θ_0 is large. Clearly the physical reason for this is that, at large angles of incidence θ_0, the boundary-layer width $\Delta\cos\theta_0$ is very small. Consequently only ray lines OP inclined at shallow angles ($\vartheta \sim 90°$) to the interface have sufficiently long segments contained within the layer in which significant contributions to the scattered field in the direction OP can be acquired.

6. Concluding remarks

The final example of §5 illustrates in a striking manner the differences to be expected between the *coherent* scattered field, which in that case would be a *specularly reflected* $O(\epsilon^2)$ wave, and the diffusely scattered random field. This casts serious doubts on the ability of renormalized wave equations for the mean field alone (such as the Dyson equation) to account satisfactorily for the properties of propagation in an extensive random medium, and indicates that great care should be exercised in the use and interpretation of such equations. This is especially true at high frequencies, and the one-dimensional transmission problem treated in §5.2 demonstrates the importance of the random scattered component of the wave field in resolving the *forward scatter* paradox.

The author considers that the kinetic theory constitutes a significant improvement on classical theories of scattering, as well as on the method based on the use of the Bethe–Salpeter equation for the propagation of correlations, which has been briefly criticized in §2. The theory satisfies the principle of conservation of total wave energy and the Second Law of Thermodynamics. It will probably lead to a revision of many of the current ideas concerning wave propagation in random media. Whenever extensive scattering regions are involved it appears that considerable divergence from classical predictions must be expected to arise.

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